Hackenbush: Blue and Red

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Introduction

Hackenbush Blue and Red is a combinatorial, partizan game with very simple rules invented by John Conway, yet a study of the game can lead to some interesting mathematics.

Each game of Hackenbush Blue and Red has a predetermined value, which can be found by computing and analyzing the simple numbers of each possible move within the game. Once a player determines the value, they can use these computations to choose the best possible moves to ensure they will win the game.

Hackenbush Blue and Red is a fun way to explore the logical thinking of mathematics and would be a great addition within mathematics classrooms. The breakdown of this game would contribute to the development of students' logical thinking and understanding of how it can be applied in everyday life.

How To Play Hackenbush

Hackenbush Blue and Red is a collection of points connected by line segments that are colored either red or blue. In every game, there will always be a minimum of one segment directly connected to the "ground level", which for the purpose of this project, will always be a horizontal black line. Also, it is important to note, a segment can be considered connected to the ground level indirectly if it is attached to another segment connected to the ground level.

Hackenbush Blue and Red consists of two players who alternate moves. Red moves by cutting a red segment and Blue, by cutting a blue one. The cut segment is deleted together with any other segments that are no longer connected to the ground. When a player is unable to move (in other words, when it is his turn and there are no longer any segments of his color), he loses.

Definitions & Theorems

Simplicity Rule:

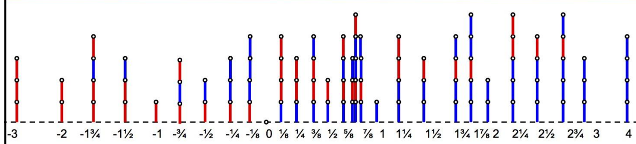
1. If there is an integer *n* satisfying $b < n < r$ the $v(G)$ is the closest integer to 0
2. Otherwise $v(G)$ is the (unique) rational number x satisfying $b < x < r$ whose denominator is the smallest possible power of 2.

Partizan Game: Each player has a different set of possible moves from a given position, in other words, each player has moves that are not allowed by their opponent.

Combinatorial Game: A game in which the following conditions are met:

1. There are two players moving alternately;
2. There are no chance devices and both player have perfect information;
3. The rules are such that the game must eventually end; and
4. There are no draws and the winner is determined by who moves last.

Hackenbush Strings



Simple Numbers

0
Integers
$^m/_2$
$^m/_4$
$^m/_8$

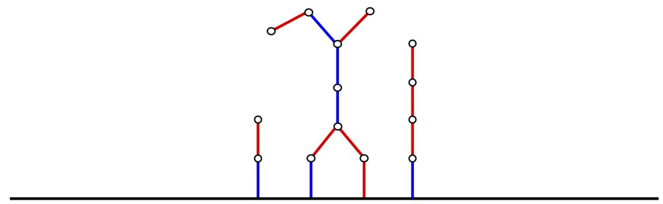
Simple numbers are used to determine the value of a hackenbush game based on the value after both blue and red make their best move. The simplest number possible is 0 and following that the next simplest number would be any integer. After the integers the next simplest number is any integer *m* over 2, $^m/_2$. The simple numbers continue by doubling the denominator as seen in the table so simple numbers continue as $^m/_4$, $^m/_8$, and so on. These numbers are used in Hackenbush to determine the value of each game. We write the value of the game as {value after blue makes its best move | value after red makes its best move}. Using both values after red and blue make their best move the value of the game can be determined by finding the simplest number between these two numbers.

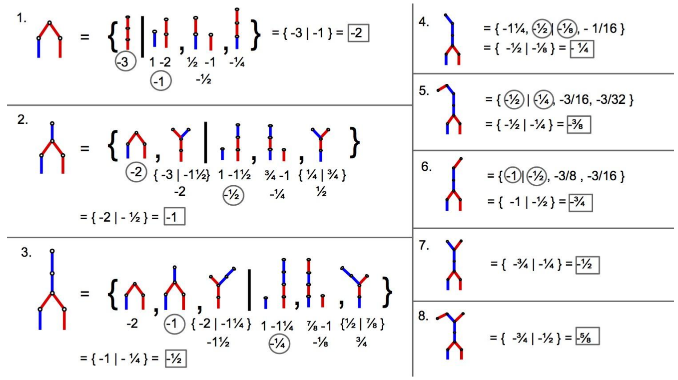
Method of Determining Game Value

There is a predetermined winner for every game of Hackenbush if you assume the winner is making their best moves. If the value of the game is positive, that means that blue will win. If the value of the game is negative, then red will win. If the game is a zero game, then whoever takes the first move will lose. So, determining the value of the game is imperative if you would like to have the upperhand. The hackenbush example on the left is an example of a zero game. Knowing the value of the game is zero lets you know that you should choose to go second, and if you do so, you will win.

To determine the value of the game, we decided to start at the bottom and work our way up. We started with a simple shape, and then added piece by piece until we had the entire figure. For every step our goal is to find {value after blue makes its best move | value after red makes its best move} so that we can then find the simple number and determine the value for that section of the game. In step 1, we laid out all of the possible moves for blue and red to make. Because there is only 1 blue move, we could determine easily that the best move is -3. We then compare all of the options for red. The values of red's moves include { -1, -½, -¼}. Red always wants to choose the greatest negative value, therefore -1 is the best move for red. We then take the simple number between {-3 | -1}, which is -2. So the value of the same game in step 1 is -2. As the we continue adding pieces to the figure, the values become more and more complicated. We repeat a similar process until we have our final value for our game, which is -⅝. We add two additional games, with the value of one game equal to ½ and the value of the other game equal to ⅛ so that we have a sum of games whose value is zero.

Hackenbush Example





Future Directions

As we have discovered, Hackenbush Blue and Red has a whole hidden background of detail oriented mathematics which allow the players to compute values in order to make the best moves to win the game. As future educators, we believe by integrating games with mathematical backgrounds into the classroom, such as Hackenbush Blue and Red, we can help students continue to develop and expand their logical thinking skills. These skills are not only important within the mathematics classroom, but also throughout various situations in everyday life. Hackenbush Blue and Red is just the tip of the iceberg when it comes to different mathematical games and as future educators we plan to continue researching other engaging mathematical resources to help our students develop critical skills and thoughts.

Sources: Bartlett, Padraic. "A Short Guide to Hackenbush." (n.d.): n. pag. 12 Aug. 2006. Web. 2 Apr. 2015. <http://www.link.cs.cmu.edu/15859-s11/notes/Hackenbush.pdf>.

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