The Arithmetic Derivative

Donald McCrae, Tyler Snyder, Tyler Vance
Advised: Dr. Stephen Gubkin

Abstract

The arithmetic derivative is a simple function defined using the unique prime factorization of integers and the product rule from calculus. This is quite deceiving, however, as the properties and behavior of the derivative are directly related to some of the oldest and most studied conjectures in elementary number theory. The arithmetic derivative operator is defined to be the unique map which sends every prime integer to 1 and which satisfies the "product rule" that for all a, b ∈ Z, (ab)' = a'b+ab'. For our research paper, we will use proof by induction on (nk)'= knk-1n' to show that it holds true for all positive integers. We hope to familiarize the reader with the notation and properties of the arithmetic derivative.

Background

The derivative is a well known math aspect that has been explored for many years. However, the arithmetic derivative allows numbers to be explored in a similar fashion as the regular derivative, just without the variable. Moreover, the arithmetic derivative has different definitions from the variable derivative. These definitions are

0'=0

Let n represent all prime numbers:

n'=1

(ab)'=ab'+a'b

OBJECTIVES

- Prove the validity of the chain rule by using proof by induction
- •Create a program to determine the arithmetic derivative of any number

METHODS

•We established a solid understanding of proofs by induction.

•This understanding allowed us to properly determine the proof on our own

•Then we created a program which allowed us to determine the arithmetic derivative of any number.

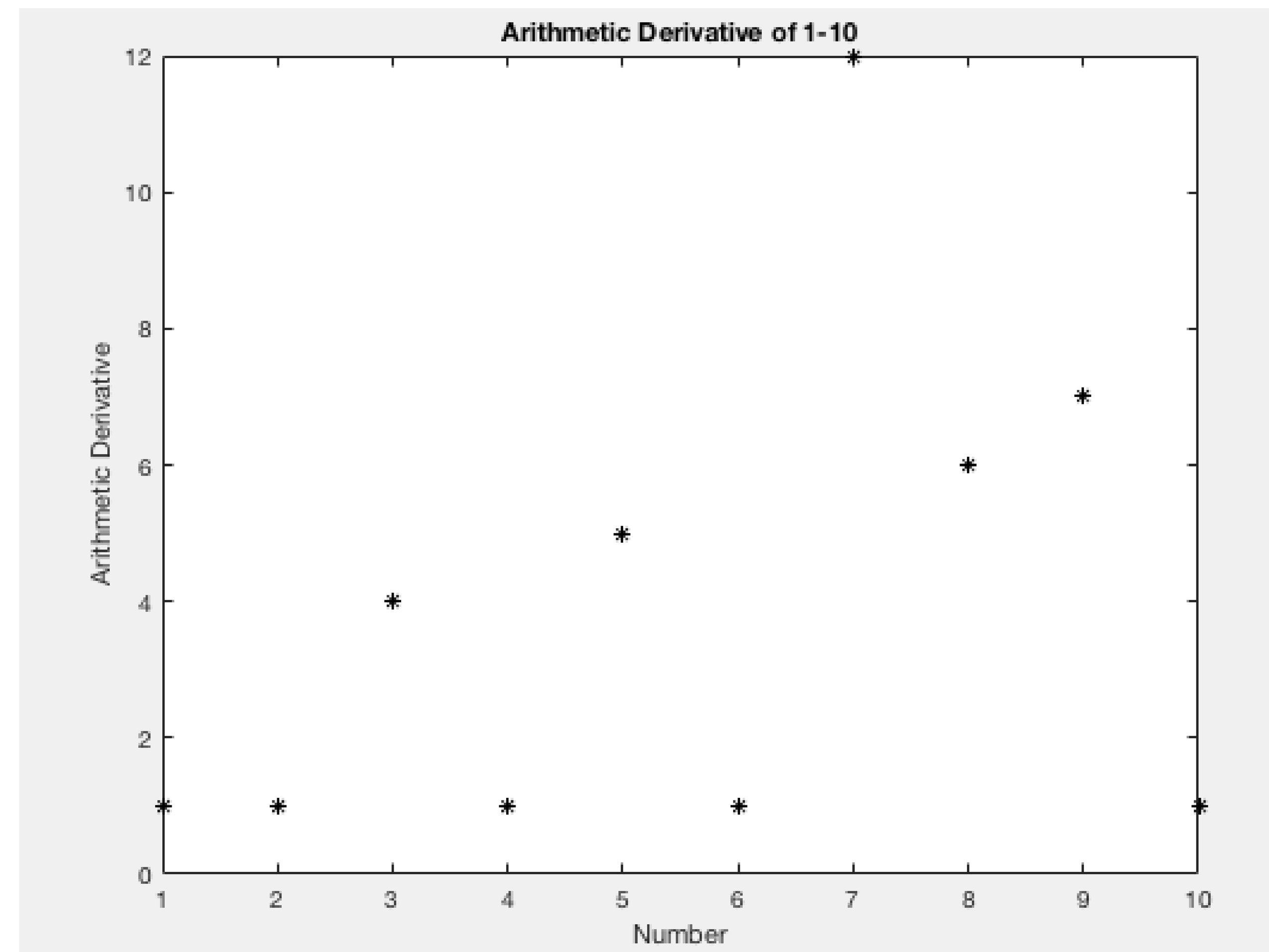

Figure 1. Arithmetic Derivative of numbers 1-10

RESULTS

- We created a program to determine the arithmetic derivative of any number using the product rule (as seen in Figure 2.
- By proof by induction, the chain rule does work for the arithmetic derivative of all natural numbers.

```
def find_factor(n):
   i = 2
   factor = 0
   while i <= n:
      if n % i:
         i += 1
      else:
         factor = i
         return factor

#print(find_factor(49))

def is_prime(n):
   return n==find_factor(n)

#print(is_prime(14))

def arithmetic_derivative(n):
   if n==0:
      return False
   if n==1:
      return 0
   if is_prime(n):
      return 1
   else:
      return
find_factor(n)*arithmetic_derivative(n/find_factor(n))+n/find_factor(n)*arithmetic_derivative(find_factor(n))

#for i in range(0,61):
#   print(arithmetic_derivative(i))

def higher_derivative(k,n):
   i=0
   d = n
   while i<k:
      d = arithmetic_derivative(d)
      i=i+1
   return d

#for i in range(0,10000):
#   if i==arithmetic_derivative(i):
#      print(i)
```

Figure 2. Arithmetic Derivative Program.

CONCLUSIONS

The chain rule is valid when using the arithmetic derivative. This showed by the proof in Figure 3. This allows us to confirm that the main ideas of a normal derivative can be applied to the arithmetic derivative.

Arithmetic Derivative: Proof by Induction

$(n^k)' = kn^{k-1}n'$ **We will prove this with induction on k**

1. Base case (Prove true for k=1)

$(n^1)' = n' = 1n^{1-1}n'$

=> n' Therefore base case holds true

2. Suppose the arithmetic derivative holds true for k= p-1

$(n^{p-1})' = (p-1)n^{p-2}n'$

3. Show the arithmetic derivative holds true for k=p

$(n^p)' = (n^{p-1}*n)' =$

$<= (n^{p-1})*n + n^{p-1}*n'$ (Induction step on n^{p-1})

$<= (p-1)*n^{p-2}*n'*n + n^{p-1}*n'$

$<= n^{p-1}*n'(P-1+1)$

$<= Pn^{p-1}*n'$

Induction is complete, hence the arithmetic derivative holds true for all positive integers

Figure 3. Chain Rule Proof By induction

FUTURE WORK

We can further experiment with the arithmetic derivative of certain numbers with the program we have created. Also, we can use proofs by induction to see if other properties of a regular derivative apply to those of the Arithmetic Derivative. These proofs will allow us to use specific examples to back up our written out work.

References

Sandhu, Alaina. *An Exploration of the Arithmetic Derivative*. pp. 1–14, *An Exploration of the Arithmetic Derivative*.

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