Desmos and Dynamics

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Dynamical Systems

A dynamical system is a mathematical model for a system that evolves in time.

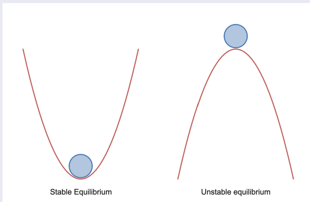
Dynamical systems are used to study complex dynamic behaviors such as sustained oscillations, phase transitions, hysteresis, coupling and chaos.

Equilibria and Stability

An equilibrium is a stationary system state. If a system is in an equilibrium state, it remains at that state unless perturbed by external influences.

An equilibrium point is stable if the system tends to return to the original state after a small perturbation.

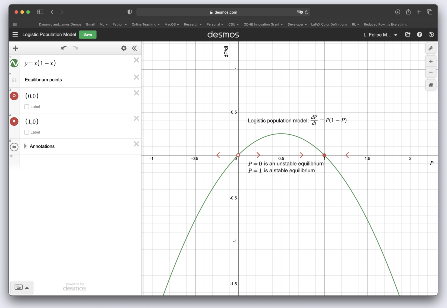
An equilibrium is unstable if small perturbations result in trajectories that move away from the original state.



Desmos

Desmos (`https://www.desmos.com`) is an online tool for exploring mathematical graphs.

Through Desmos, students can experiment with graphical analysis of one-dimensional dynamical systems.



(`https://www.desmos.com/calculator/gxuikatuyb`)

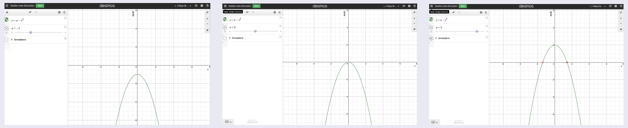
Bifurcations

Parameters are used to represent details of the environment where a dynamical system evolves.

As parameters change, equilibrium points can be created, destroyed and/or change stability.

Desmos sliders provide a tool to analyze system behavior as parameters change.
Example:

$$\frac{dx}{dt} = a - x^2$$



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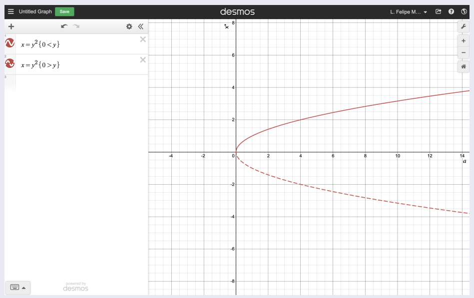
No equilibrium points if $a < 0$.
One equilibrium point at $x = 0$ if $a = 0$.
Two equilibrium points if $a > 0$: $x_1^* = -\sqrt{a}$ is unstable and $x_2^* = \sqrt{a}$ is stable.

A saddle-node bifurcation happens at the critical parameter value $a_c = 0$. In this kind of bifurcation, a pair of equilibrium points is created/destroyed. One of the equilibria is stable, and the other is unstable.

Bifurcation Diagrams

A bifurcation diagram has the parameter values on the horizontal axis and corresponding equilibrium points on the vertical axis.
Stable equilibria are represented by solid curves, and unstable equilibria by dashed curves.



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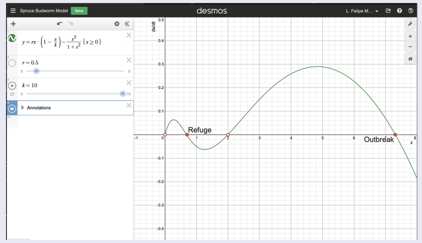
Spruce Budworm Model

The spruce budworm is a parasite of the Canadian fir tree. Outbreaks of the budworm can devastate whole forests.

Ludwig et al. (1978) proposed the following model for the budworm population $x(t)$:

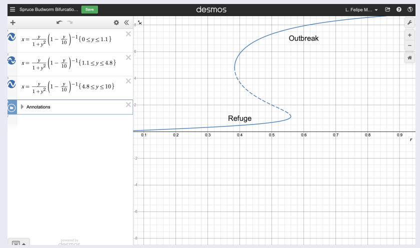
$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}$$



(`https://www.desmos.com/calculator/ktngyq565m`)

There can be 1, 2 or 3 positive equilibria. In the cases where there are three equilibria, the smallest equilibrium represents the refuge level of the population, and the largest equilibrium represents the outbreak level.

Hysteresis



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From the bifurcation diagram, the following can be concluded:
For large k, the system has two saddle-node bifurcations.
If r is small, the refuge level is a stable equilibrium.
If r grows beyond the first saddle-node bifurcation, the refuge equilibrium disappears, and the population size rapidly converges to the outbreak level.
Even if r returns to its original value, the budworm population does not return to the refuge level. This phenomenon is called hysteresis.