THE SECOND ANNUAL CSU FRESHMAN–SOPHOMORE MATHEMATICS COMPETITION

SOLUTIONS

April 30, 2008

(1) Four friends bought a boat. The first friend paid one half of the total paid by the other three; the second paid one third of the total paid by the other three; the third paid one quarter of the total paid by the other three; and the fourth paid \$130. How much did the boat cost, and how much did each of the friends pay?

Solution: Let p_i denote the amount payed by the i-th friend. We obtain a linear system:

$$\begin{aligned} p_1 &= \tfrac{1}{2}(p_2 + p_3 + p_4) \\ p_2 &= \tfrac{1}{3}(p_1 + p_3 + p_4) \\ p_3 &= \tfrac{1}{4}(p_1 + p_2 + p_4) \\ p_4 &= 130. \end{aligned}$$

Solving this system we get $p_1 = 200$, $p_2 = 150$, $p_3 = 120$, $p_4 = 130$. The boat cost \$600.

(2) Let a be the sum of three consecutive integers and let b be the sum of the *next* three consecutive integers. Is it possible that $ab = 111111111$?

Solution: If $ab = 111111111$ then neither a nor b is even. But if a is odd, it equals the sum of two even and one odd number. In this case b is the sum of two odd and one even number, since b be the sum of the *next* three consecutive integers. Therefore, b is even. This contradiction shows that $ab = 111111111$ is impossible.

(3) Let ABC be a right triangle with hypotenuse AB and angle $A = 30°$. Let M be the midpoint of AB and let N be a point on AC such that MN is perpendicular to AB. Find the length of MN if the length of AC equals 10.

Solution: One possible solution is to notice that triangle ABC is similar to triangle ANM. Hence $NM/BC = AM/AC$. Also $AM = BC$ since $BC/AB = \sin(30°) = 1/2$ and $AB = 2AM$. Therefore we have

$$\frac{NM}{BC} = \frac{BC}{10} = \tan(30°) = \frac{1}{\sqrt{3}}.$$

This implies that $NM = 10/3$.

Here is another one: Let ML be the median in the triangle AMN. Then $AL = LN = ML$ since AMN is a right triangle. On the other hand $\angle LNM = \angle LMN = 60°$. Therefore $NM = ML = NL$. Similarly $\angle NMC = \angle NCM = 30°$ and, hence, $NM = NC$. We have shown that NM equals a third of AC, i.e. $10/3$.

(4) Find *all* possible integers p and q such that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{pq}=1.$$

Solution: Clearing the denominators and rearranging the terms we have $pq-p-q=1$. This can be written as a product $(p-1)(q-1)=2$. Now both $p-1$ and $q-1$ are integers and divisors of 2. We have four cases: (1) $p-1=-1$ and $q-1=-2$; (2) $p-1=1$ and $q-1=2$. The other two are obtained by switching p and q. Note that case (1) is impossible since $p\neq 0$ in the original equation. Therefore, $p=2,\ q=3$ and $p=3,\ q=2$ are the only possible solutions.

(5) Let $f(x)=x^{2n}-nx^{n+1}+nx^{n-1}-1$ for some $n>1$. Show that $x=1$ is a root of $f(x)$. Find the multiplicity of this root.

Solution: The first part is immediate: $f(1)=1-n+n-1=0$. To find the multiplicity k of the root we use the following fact: k is the largest integer such that 1 is a root of the $k-1$-th derivative of f. We have:

$$f'(x)=2nx^{2n-1}-n(n+1)x^n+n(n-1)x^{n-2},$$

$$f'(1)=2n-n(n+1)+n(n-1)=0;$$

$$f''(x)=2n(2n-1)x^{2n-2}-n^2(n+1)x^{n-1}+n(n-1)(n-2)x^{n-3},$$

$$f''(1)=2n(2n-1)-n^2(n+1)+n(n-1)(n-2)=0.$$

Finally,

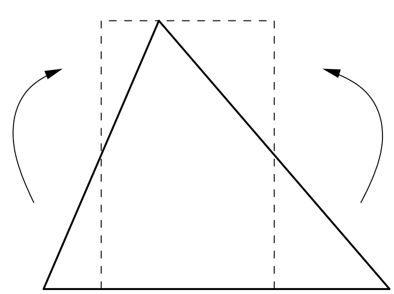
$$f'''(x)=2n(2n-1)(2n-2)x^{2n-3}-n^2(n+1)(n-1)x^{n-2}+n(n-1)(n-2)(n-3)x^{n-4},$$

$$f'''(1)=2n(2n-1)(2n-2)-n^2(n+1)(n-1)+n(n-1)(n-2)(n-3)=2n^3-2n.$$

This calculation shows that the multiplicity is 3 for all $n>3$. The cases $n=2$ and $n=3$ can be treated separately. In each case one can see that the multiplicity equals 3.

(6) Show how to cut an arbitrary triangle into three pieces which, after rearranging, form a rectangle.

Solution:



(7) Evaluate the integral

$$\int_0^{\pi}\left(|\sin(2007x)|-|\sin(2008x)|\right)\,dx.$$

Solution: Let us compute each integral separately. In the first integral after the substitution $u = 2007x$ we get

$$\frac{1}{2007}\int_0^{2007\pi} |\sin u|\, du.$$

Note that the integral of $|\sin u|$ over every interval $[(i-1)\pi, i\pi]$ equals 2. Therefore the above integral equals

$$\frac{1}{2007}\int_0^{2007\pi} |\sin u|\, du = \frac{1}{2007}(2\cdot 2007) = 2.$$

Similarly, the second integral in the question equals 2 and their difference is zero.

(8) Let $f(x) = x + \frac{1}{x}$, $g(x) = x^2$, and $h(x) = (x-1)^2$. You are allowed to take sums and products of the above functions, as well as to multiply by a constant or add a constant to any of them (in any order). The same operations can be applied to functions you obtain.
 (a) Obtain $1/x$ using the above operations.
 (b) Is it possible to obtain $1/x$ starting with $g(x)$ and $h(x)$ only? Explain.
 (c) Is it possible to obtain $1/x$ starting with $f(x)$ and $h(x)$ only? Explain. (Hint: Look at the derivative at $x = 1$.)

Solution:
(a) First let's obtain x. We have $h(x) = (x-1)^2 = x^2 - 2x + 1$. Thus $g(x) - h(x) = 2x - 1$ and we get $x = \frac{1}{2}(g(x) - h(x) + 1)$. Now we just need to subtract this from $f(x)$:

$$\frac{1}{x} = f(x) - \frac{1}{2}(g(x) - h(x) + 1).$$

(b) No it is impossible, since both $g(x)$ and $h(x)$ are polynomials and the allowed operations can only produce more polynomials.
(c) Note that $f'(1) = 0$ and $h'(1) = 0$. Moreover, if two functions have derivative zero at $x = 1$ then the same is true about their sum and their product. For the sum it is obvious, for the product it follows from the "product rule":

$$(f_1 f_2)' = f_1' f_2 + f_1 f_2', \quad \text{so} \quad (f_1 f_2)'(1) = 0\cdot f_2(1) + f_1(1)\cdot 0 = 0.$$

Also rescaling a function or adding a constant to it does not change the property of having zero derivative at $x = 1$. Therefore, any function obtained using the allowed operations also has zero derivative at $x = 1$. But the derivative of $1/x$ at $x = 1$ is not zero, therefore it cannot be obtained using the allowed operations.

(9) Peter tossed a coin 5 times and Mary tossed a coin 6 times. Each of them recorded how many times their tosses were "heads". What are the chances that Mary's number is strictly greater than Peter's?

Solution: A straightforward way would be to compute all possible outcomes of Peter's 5 tosses and Mary's 6 tosses where Mary's number is strictly greater than Peter's. If all Peter's tosses were "tails" any outcome of Mary's tosses works except for when all her tosses were also "tails", which is $2^6 - 1$. Then there are 5 outcomes when exactly one Peter's toss was "heads" in which case all but 7 Mary's tosses work, etc. We obtain

$$\binom{5}{0}\left(2^6 - \binom{6}{0}\right) + \binom{5}{1}\left(2^6 - \binom{6}{1} - \binom{6}{0}\right) + \cdots + \binom{5}{5}\left(2^6 - \binom{6}{5} - \cdots - \binom{6}{0}\right).$$

Simplifying this (which is not pleasant!) we get 2^{10}. This is the number of favorable outcomes out of all possible $2^5 \cdot 2^6 = 2^{11}$. Therefore the probability is $1/2$.

To avoid these tedious calculations let's notice that to every favorable outcome we can associate an unfavorable. Indeed, suppose we have 5 Peter's tosses and 6 Mary's tosses and let h_P and h_M denote how many tosses were "heads" in Peter's and in Mary's tosses, respectively. If $h_P < h_M$, this is a favorable outcome. Now flip all Peter's tosses ("heads" to "tails" and vise versa) and all Mary's tosses. In this new outcome we have $5 - h_P$ "heads" in Peter's tosses and $6 - h_M$ in Mary's. Clearly, $5 - h_P \geq 6 - h_M$ (since $h_P \leq h_M - 1$), i.e. this is an unfavorable outcome, associate with the original favorable. Therefore, the number of favorable outcomes equals the number of unfavorable ones, hence, the probability is $1/2$.

(10) Does there exist a positive integer which is a multiple of 2008 and whose only digits (in decimal representation) are 0 and 1?

Solution: The answer is yes, and here is how we can see that. Consider a sequence of numbers all whose digits are 1:

$$1,\ 11,\ 111,\ 1111,\ 11111,\ \ldots\ , 111\cdots1111,$$

where the last number has 2009 ones. Since there are 2009 numbers total, there exit two of them with the same remainder modulo 2008. Their difference is then a multiple of 2008. On the other hand their difference is clearly an integer containing digits 0 and 1 only.