THE FIRST ANNUAL CSU FRESHMAN–SOPHOMORE MATHEMATICS COMPETITION

SOLUTIONS

April 30, 2007

(1) Alice wrote down the numbers from 1 to 500 on the board and got a huge integer:

$$123456\ldots498499500.$$

Jack erased the first 500 digits. What is the first digit of the remaining integer?

Solution: The first 9 digits are the numbers from 1 to 9. We have $99-9=90$ two digit integers (from 10 to 99). They take up the next 180 digits. Now 100 three digit integers from 100 to 199 take up another 300 digits. We have $500-9-180-300=11$ digits left to erase. They are

$$20020120220$$

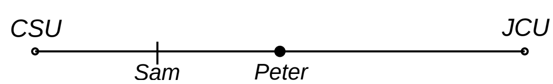
and the next digit is 3.

(2) Peter and Sam were biking from CSU to John Carroll. Sam was going with a constant speed. Peter was going twice as fast as Sam for the first half of the distance, and half as fast as Sam for the rest of the distance. Who arrived at John Carroll first?

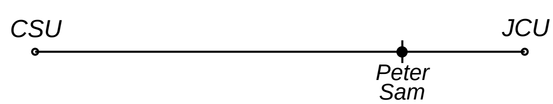
Solution: Notice that for the second half of the distance Peter was going twice as slow as Sam. That is exactly the time needed for Sam to travel the whole distance. Since Peter also spent some time on the first half of the distance he arrived second.

Here is another nice pictorial solution:

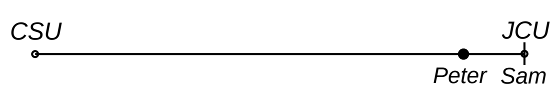
When Peter reaches the midpoint, Sam has gone only half as far:



From this moment on, however, Sam is travelling twice as fast as Peter, so Sam covers half the distance while Peter covers the next quarter:



And when Sam reaches John Carroll, Peter's gone only half the remaining distance:



(3) Find the last digit in 13^{2007}.

Solution: To find the last digit in 13^{2007} we need to see how the last digit changes when we take successive powers of 3. We have $3^0 = 1,\ 3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81$. Notice that the last digit cycles modulo 4:

$$1, 3, 9, 7, 1, 3, 9, 7, \ldots$$

Now 2007 has remainder 3 modulo 4 since $2007 = 501 \cdot 4 + 3$. Hence the last digit of 3^{2007} equals the last digit of 3^3, which is 7. Therefore, the last digit in 13^{2007} is 7.

(4) Let a and b be integers. Is it true that if a^2 is divisible by $a + b$ then so is b^2? Explain your answer.

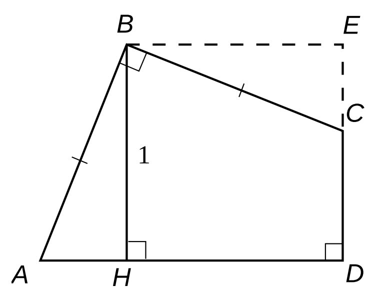
Solution: Yes, it is true. Recall that $a^2 - b^2 = (a + b)(a - b)$. Therefore, if $a^2 = (a + b)m$ for some m then

$$b^2 = a^2 - (a + b)(a - b) = (a + b)m - (a + b)(a - b) = (a + b)(m - a + b),$$

i.e. b^2 is also divisible by $a + b$.

(5) The quadrilateral below has two right angles B and D and two congruent sides AB and BC. Find the area of the quadrilateral if the height BH equals 1.

Solution: Let E be the intersection point of the line DC and the line through B parallel to AD.

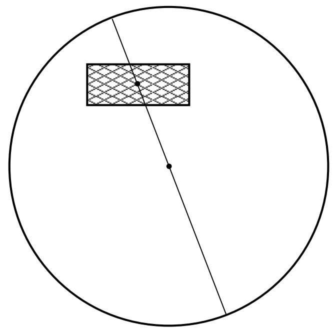


We will show that the two triangles $\triangle ABH$ and $\triangle CBE$ are congruent. Indeed, clearly $\angle BHA = \angle BEC = 90°$. Also $\angle ABH = \angle ABC - \angle HBC = 90° - \angle HBC$ and similarly $\angle CBE = \angle HBE - \angle HBC = 90° - \angle HBC$. This, together with $|AB| = |CB|$, shows that the triangles are congruent.

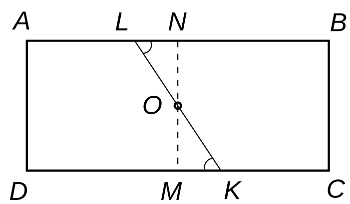
Now, since $\triangle ABH$ and $\triangle CBE$ are congruent, we have $|BH| = |BE| = 1$, so $HBED$ is a unit square. Also, the area of the quadrilateral $ABCD$ equals the area of the square $HBED$, which is 1.

(6) The twin brothers Billy and Willy got a cake for their birthday. The cake has a chocolate bar on top (not necessarily in the center). How can they cut the cake in half so that Billy and Willy get the same amount of chocolate? Explain your answer.

Solution: They have to cut along the line through the center of the cake and the center of the chocolate bar.



Indeed, it is clear that any line through the center of the cake cuts it in half. Let us show that any line through the center of a rectangle (chocolate bar) divides it into two congruent trapezoids.



Let O be the center of the rectangle $ABCD$. We have $AB \,||\, DC$, so $\angle OLN = \angle OKM$. Also $|OM| = |ON|$. Thus the triangles $\triangle OLN$ and $\triangle OKN$ are congruent, so $|LN| = |KM|$. This implies that $|LB| = |KD|$ and also $|CK| = |AL|$. Therefore the trapezoids $CKLB$ and $ALKD$ are congruent.

(7) Consider the function

$$f(x) = \frac{x}{\sqrt{1+x^2}}.$$

Find the 2007-th interation of the composition of f with itself:

$$\underbrace{f(f(f(\dots f(x)\dots)))}_{\text{2007 times}}.$$

Solution: Taking one composition we get

$$f(f(x)) = \frac{f(x)}{\sqrt{1+f(x)^2}} = \frac{x/\sqrt{1+x^2}}{\sqrt{1+x^2/(1+x^2)}} = \frac{x/\sqrt{1+x^2}}{\sqrt{1+2x^2}/\sqrt{1+x^2}} = \frac{x}{\sqrt{1+2x^2}}.$$

Iterating this several times one can notice that every time the coefficient of x^2 under the radical increases by 1. Therefore, the formula should be

$$\underbrace{f(f(f(\dots f(x)\dots)))}_{\text{2007 times}} = \frac{x}{\sqrt{1+2007x^2}}.$$

To have a formal proof one could argue by induction.

The base of induction is trivial, so assume the formula for the k-th iteration is

$$\underbrace{f(f(f(\dots f(x)\dots)))}_{k \text{ times}} = \frac{x}{\sqrt{1+k\,x^2}}.$$

Applying f to both sides of this equation we get

$$\underbrace{f(f(f(\dots f(x)\dots)))}_{k+1 \text{ times}} = f\left(\frac{x}{\sqrt{1+k\,x^2}}\right) = \frac{x/\sqrt{1+k\,x^2}}{\sqrt{1+x^2/(1+k\,x^2)}}$$

$$= \frac{x/\sqrt{1+k\,x^2}}{\sqrt{1+(k+1)x^2/\sqrt{1+k\,x^2}}} = \frac{x}{\sqrt{1+(k+1)x^2}},$$

i.e. the formula for the $(k+1)$-st iteration follows. By the principle of math induction the formula holds for any number of iterations.

(8) Three numbers k, l, and m satisfy $k+l+m=0$ and $kl+km+lm=0$. Find the numbers.

Solution: We will show that $k=l=m=0$. There are many ways to do this, here is one of the most elegant solutions. We have

$$0=(k+l+m)^2=k^2+l^2+m^2+kl+km+lm=k^2+l^2+m^2.$$

But in the latter each summand is non-negative, so their sum is zero if and only if each of them is zero. Therefore $k^2=l^2=m^2=0$ which implies $k=l=m=0$.

(9) Let $f(x)=ax^2+bx+c$ be a quadratic polynomial with no real roots. If the sum of its coefficients $a+b+c$ is negative, what can you say about the sign of c?

Solution: Since $f(x)$ has no real roots, its graph (parabola) either lies strictly above the x-axis (and then all the values of $f(x)$ are positive) or below the x-axis (and then all the values of $f(x)$ are negative). We know that $f(1)$ is negative since $f(1)$ is precisely $a+b+c$. Therefore, $f(x)$ is negative for all x. In particular $f(0)=c$ is negative.

(10) The four roots of the polynomial $p(x)=x^4-10x^2+a$ form an arithmetic sequence (i.e. the distance between any two adjacent roots is the same). Find a.

Solution: Let $r<s<u<v$ be the four roots of the polynomial $p(x)$. Notice that all the exponents of x in $p(x)$ are even so $p(x)=0$ implies $p(-x)=0$. Therefore, the four roots are symmetric with respect to the origin:

$$-v, \quad -u, \quad u, \quad v.$$

If the roots form an arithmetic sequence we must have $v-u=u-(-u)$. This implies that $v=3u$ and so the four roots are

$$-3u, \quad -u, \quad u, \quad 3u.$$

Now we get the factorization of $p(x)$ into four linear terms:

$$x^4-10x^2+a=(x+3u)(x+u)(x-u)(x-3u).$$

To find a we simply expand the right hand side and equate the coefficients:

$$x^4-10x^2+a=(x^2-(3u)^2)(x^2-u^2)=x^4-10u^2x^2+9u^4.$$

We see that $-10=-10u^2$ and $a=9u^4$. This implies that $u^2=1$ and hence $a=9$ (and the roots are $-3,-1,1,3$).

(11) Consider the function

$$f(x) = \frac{|x-1|}{|x+1|}.$$

Find its absolute maximum and absolute minimum on the segment $[-\frac{1}{2}, 2]$. Explain your answer.

Solution: First, $f(x) \geq 0$ for all x and $f(x) = 0$ for $x = 1$ only. Therefore the absolute minimum of $f(x)$ is 0 and is attained at $x = 1$.

For the absolute maximum, note that on the segment $[-\frac{1}{2}, 2]$ we have $x+1 > 0$, hence

$$f(x) = \begin{cases} -\frac{x-1}{x+1}, & \text{for } -\frac{1}{2} \leq x < 1 \\ \frac{x-1}{x+1}, & \text{for } \quad 1 \leq x \leq 2. \end{cases}$$

Next

$$-\frac{x-1}{x+1} = -1 + \frac{2}{x+1} \quad \text{and} \quad \frac{x-1}{x+1} = 1 - \frac{2}{x+1},$$

which shows that $f(x)$ strictly decreases on $[-\frac{1}{2}, 1]$ and strictly increases on $[1, 2]$. (This can also be shown by taking the derivatives.) Therefore the absolute maximum is attained at one of the endpoints. Since $f(-\frac{1}{2}) = 3$ and $f(2) = \frac{1}{3}$ the absolute maximum is 3 attained at $x = -\frac{1}{2}$.

(12) Evaluate the integral

$$\int_0^{\ln 2} \sqrt{e^x - 1}\, dx.$$

Solution: We make a substitution $u = \sqrt{e^x - 1}$ (there are other substitutions that work as well). Then $e^x = u^2 + 1$ and $e^x\, dx = 2u\, du$, so

$$dx = \frac{2u}{e^x}\, du = \frac{2u}{u^2+1}\, du.$$

Also $x = 0$ becomes $u = \sqrt{e^0 - 1} = 0$ and $x = \ln 2$ becomes $u = \sqrt{e^{\ln 2} - 1} = 1$. Now we have

$$\int_0^{\ln 2} \sqrt{e^x - 1}\, dx = \int_0^1 \frac{2u^2}{u^2+1}\, du = \int_0^1 \left(2 - \frac{2}{u^2+1}\right) du$$
$$= 2 - 2 \arctan u|_0^1 = 2 - \frac{\pi}{2}.$$