

THE FIFTH ANNUAL CSU FRESHMAN–SOPHOMORE
MATHEMATICS COMPETITION

SOLUTIONS

- (1) Let $\{a_n\}$ be an arithmetic sequence (i.e. $a_n = c + nq$ for some constants c and q). It is known that

$$a_{1011} = 2011 \quad \text{and} \quad a_{2011} = 1011.$$

Compute a_{3022} . (Give a numerical answer if possible.)

Solution: Substituting $n = 1011$ and $n = 2011$ into the general formula for a_n we get a system of two linear equations $2011 = c + 1011q$ and $1011 = c + 2011q$. The solution is $q = -1$ and $c = 3022$. Thus $a_{3022} = c + 3022q = 3022 + 3022(-1) = 0$.

- (2) Let a , b , and c be real numbers. Consider the quadratic polynomial

$$f(x) = x^2 - (a + b)x + (ab - c^2).$$

Show that both a and b lie between the roots of the polynomial $f(x)$.

Solution: The graph of $f(x)$ is a parabola whose branches point up. Therefore, $x = a$ lies between the roots of $f(x)$ if and only if the value of $f(x)$ at $x = a$ is less than or equal to zero. Here we have $f(a) = a^2 - (a + b)a + (ab - c^2) = -c^2 \leq 0$. Similarly, for $x = b$.

- (3) Let O_n denote the sum of the first n positive odd squares, and let E_n denote the sum of the first n positive even squares. For example,

$$O_3 = 1^2 + 3^2 + 5^2 = 35 \quad \text{and} \quad E_3 = 2^2 + 4^2 + 6^2 = 56.$$

Find a general formula for their quotient O_n/E_n as a function of n . You may find it useful to know that the sum of the squares of the first n positive integers is

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: First, note that $O_n + E_n$ is the sum of the squares of the first $2n$ positive integers, hence,

$$O_n + E_n = 1^2 + 2^2 + \cdots + (2n)^2 = \frac{2n(2n+1)(4n+1)}{6}.$$

Second,

$$E_n = 2^2 + 4^2 + \cdots + (2n)^2 = 4(1^2 + 2^2 + \cdots + n^2) = \frac{4n(n+1)(2n+1)}{6}.$$

Therefore,

$$\frac{O_n}{E_n} + 1 = \frac{O_n + E_n}{E_n} = \frac{2n(2n+1)(4n+1)}{6} \frac{6}{4n(n+1)(2n+1)} = \frac{4n+1}{2(n+1)}.$$

This implies that

$$O_n/E_n = \frac{4n+1}{2(n+1)} - 1 = \frac{2n-1}{2n+2}.$$

- (4) Show the following inequalities:

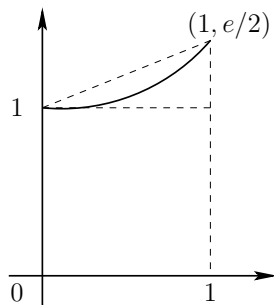
$$1 \leq \int_0^1 \frac{e^x}{1+x} dx \leq \frac{e+2}{4}.$$

(Hint: What can you say about the concavity of the function under the integral?)

Solution: The idea is to notice that the function $f(x) = \frac{e^x}{1+x}$ is increasing and concave up on the interval $0 \leq x \leq 1$. Indeed,

$$f'(x) = \frac{xe^x}{(1+x)^2} \geq 0, \quad f''(x) = \frac{(1+x^2)e^x}{(1+x)^3} > 0,$$

for all $0 \leq x \leq 1$. This implies that the minimum of $f(x)$ on $0 \leq x \leq 1$ is $f(0) = 1$ and the maximum is $f(1) = e/2$. Hence, we have the following picture:



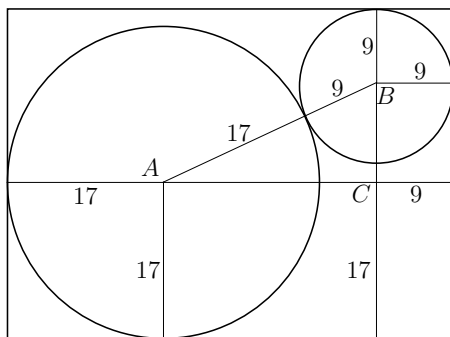
Now it is easy to see that the region below the graph of $f(x)$ on $0 \leq x \leq 1$ contains the unit square, and we obtain the first inequality. Furthermore, the region below the graph of $f(x)$ on $0 \leq x \leq 1$ is contained in the trapezoid whose area equals

$$1 + \frac{1}{2}(e/2 - 1) = \frac{e+2}{4},$$

and the second inequality follows.

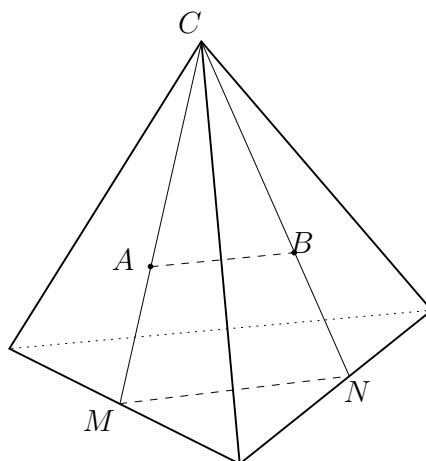
- (5) Two circles of radii 9 and 17 inches are enclosed in a rectangle, one of whose sides is 50 inches long. The two circles are tangent to each other, and each is tangent to two adjacent sides of the rectangle, as shown in the rough sketch below. What is the area of the rectangle?

Solution: Consider the right triangle ABC in the picture. Since the horizontal side of the rectangle has length 50, we have $|AC| = 50 - 17 - 9 = 24$. Also $|AB| = 17 + 9 = 26$. Therefore, $|BC| = \sqrt{26^2 - 24^2} = 10$. Now the vertical side of the rectangle has length $17 + |BC| + 9 = 36$, and, hence, the area of the rectangle is $50 \cdot 36 = 1800$ square inches.



- (6) The centers of the faces of a regular tetrahedron are the vertices of a smaller regular tetrahedron. The smaller tetrahedron has what fraction of the volume of the larger one?

Solution: For simplicity we will assume that the edge length of the tetrahedron equals one inch. Let C be the vertex and A and B be the centers of two faces of the tetrahedron as in the picture below.



Since A is, in particular, the point of intersection of the medians of that face we have $|CA|/|CM| = 2/3$. Similarly, $|CB|/|CN| = 2/3$. Furthermore, since MN joins the midpoints of the two edges, we have $|MN| = 1/2$ inch, half the size of the edge length. Now from the similar triangles CAB and CMN we get $|AB|/|MN| = |CA|/|CM| = 2/3$, so $|AB| = (2/3)|MN| = 1/3$ inch. This is the edge length of the smaller tetrahedron. It's volume is, therefore, $(1/3)^3 = 1/27$ of the volume of the larger tetrahedron.

- (7) Can you find

- (a) four
(b) five

distinct positive integers such that the sum of any three of them is a prime number?

Solution: For part (a) after a few tries one can come up with, for example, $\{1, 3, 7, 9\}$. For part (b), we will show that no such five integers exist. First, notice that if there are three positive integers whose remainders mod 3 are all different then their sum is a multiple of 3 (greater than 3) and, hence, not a prime. Therefore, only two possible remainders mod 3 may appear among the five integers. But then at least three of them will have the same remainder mod 3 and again their sum will be a multiple of 3, greater than 3.

- (8) Peter and Jonathan play a game of “one-eyed king”. They have a 6×8 “chessboard” and a king which is placed at the lower left corner of the board at the beginning of the game. Peter and Jonathan take turns moving the king to the right by one, or up by one, or diagonally up and right by one. Whoever cannot make a move loses. Peter goes first. Who has a winning strategy? Describe the strategy.

Solution: We claim that Peter has the winning strategy. The easiest way to see it is to start from the end of the game. In the picture below the squares marked by “W” are the ones Peter has to go to in order to win.

	W		W		W		W
	W		W		W		W
	W		W		W		W

Indeed, if Peter places the king at the upper right corner then Jonathan loses. Now, look at the L-shape of W's around the upper right corner. If Peter places the king in any of those squares then Jonathan will be forced to move one step away from the upper right corner, so Peter will win. Continuing in the same way we construct the larger L-shape of W's and the vertical row of W's. Since Peter goes first he will be able to put the king on a W on his first and every move.