The Third Annual CSU Freshman–Sophomore Mathematics Competition

SOLUTIONS

(1) Peter, John, and Greg are running laps. Each of them is running with a constant speed. Every ten minutes Peter passes John and every 15 minutes John passes Greg. How often does Peter pass Greg?

Solution: First, Peter's speed relative to John's is 1 lap per 1/6 hour. Next, John's speed relative to Greg's is 1 lap per 1/4 hour. Thus Peter's speed relative to Greg's is

$$\frac{1}{1/6}+\frac{1}{1/4}=\frac{1}{1/10},$$

i.e. 1 lap per 1/10 hour. Therefore, Peter passes Greg every 6 minutes.

(2) Find all values of x satisfying

$$2\sin^3(x)-\sin^2(x)-2\sin(x)+1=0.$$

Solution: First, we need to solve the cubic equation:

$$2z^3-z^2-2z+1=0,$$

where $z=\sin(x)$. It can be easily factored:

$$z^2(2z-1)-(2z-1)=(z^2-1)(2z-1)=(z-1)(z+1)(2z-1).$$

Now we need to solve three equations:

$$\sin(x)=1,\quad \sin(x)=-1,\quad \sin(x)=\frac{1}{2}.$$

The first two are satisfied when $x=\frac{\pi}{2}+\pi k$, for any integer k. The third is satisfied when $x=\frac{\pi}{6}+2\pi k$ and $x=\frac{5\pi}{6}+2\pi k$, for any integer k. Therefore the set of all x satisfying the equation in question is the union of these three infinite sets:

$$\left\{\frac{\pi}{2}+\pi k \mid k\in\mathbb{Z}\right\}\cup\left\{\frac{\pi}{6}+2\pi k \mid k\in\mathbb{Z}\right\}\cup\left\{\frac{5\pi}{6}+2\pi k \mid k\in\mathbb{Z}\right\}.$$

(3) Is it possible to divide a 14×14 grid square into grid rectangles of sizes 2×5 and 3×9? If yes, show how. If no, explain why not.

Solution: No, this is not possible. Here is one of many reasons: Suppose that we were able to do so. Denote by a and b the number of 2×5 and 3×9 rectangles, respectively. Then equating the total number of grid squares we get the equality:

$$196=10a+27b.$$

Note that $10a$ does not contribute to the last digit of $10a+27b$, so $27b$ must end with a 6. The smallest such b is 8. But this is already too big: $27\cdot 8=216$. Therefore $196=10a+27b$ is impossible for integer a and b.

(4) Find the following sum if it converges or explain why it diverges.

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n}\right).$$

Solution: By definition the sum of a series is the limit of the sequence of partial sums:

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n}\right) = \lim_{N\to\infty} \sum_{n=2}^{N} \ln\left(1-\frac{1}{n}\right).$$

Let's rewrite a partial sum using the property of the logarithm:

$$\ln\left(1-\frac{1}{2}\right)+\ln\left(1-\frac{1}{3}\right)+\cdots+\ln\left(1-\frac{1}{N}\right) = \ln\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{N}\right)\right].$$

Now we will simplify the product:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{N}\right) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{N-1}{N}\right).$$

Note that in the last expression all terms but 1 and N cancel out. Therefore the product equals $\frac{1}{N}$ and so the partial sum equals:

$$\sum_{n=2}^{N} \ln\left(1-\frac{1}{n}\right) = \ln\frac{1}{N} = -\ln N.$$

Clearly, the limit $\lim_{N\to\infty}(-\ln N)$ does not exist, so the series diverges.

(5) Compute the following sum:

$$\left\lfloor\frac{1}{3}\right\rfloor+\left\lfloor\frac{2}{3}\right\rfloor+\left\lfloor\frac{4}{3}\right\rfloor+\left\lfloor\frac{8}{3}\right\rfloor+\cdots+\left\lfloor\frac{2^{2009}}{3}\right\rfloor.$$

Here $\lfloor x\rfloor$ denotes the largest integer which is less than or equal to x (the floor function).

Solution: One possible solution is to notice that the sum of two consecutive terms (without the floor) is an integer:

$$\frac{2^i}{3}+\frac{2^{i+1}}{3} = \frac{2^i+2^{i+1}}{3} = \frac{2^i(1+2)}{3} = 2^i.$$

Here is a general property of the floor function: if $x+y$ is an integer then $\lfloor x\rfloor+\lfloor y\rfloor$ equals $x+y-1$. Indeed, $x=\lfloor x\rfloor+\{x\}$ and $y=\lfloor y\rfloor+\{y\}$ (here $\{x\}$ denotes the fractional part of x), so

$$x+y=\lfloor x\rfloor+\lfloor y\rfloor+\{x\}+\{y\}.$$

Since $0\le\{x\}<1$ and $0\le\{y\}<1$ and their sum $\{x\}+\{y\}=(x+y)-\lfloor x\rfloor+\lfloor y\rfloor$ is an integer we must have $\{x\}+\{y\}=1$. This proves the desired property.

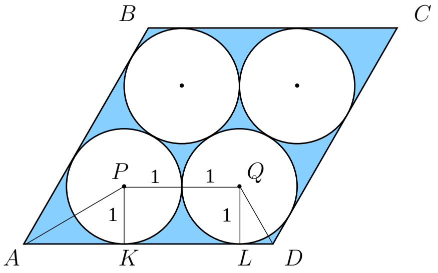
In the view of the above, we can replace every pair in our original sum with the same expression without the floor, decreased by 1:

$$\left(\frac{1}{3}+\frac{2}{3}-1\right)+\left(\frac{4}{3}+\frac{8}{3}-1\right)+\cdots+\left(\frac{2^{2008}}{3}+\frac{2^{2009}}{3}-1\right).$$

It remains to simplify this expression:

$$\frac{1}{3}\left(1+2+4+\cdots+2^{2009}\right)-1004 = \frac{2^{2010}-1}{3}-1004.$$

(6) Find the area of the gray part in the figure below, if all the circles have radius 1:



Solution: First, the centers of three mutually tangent circles form an equilateral triangle with two sides parallel to the sides of the parallelogram. Thus the angles of the parallelogram are 60° and 120°. Also by symmetry the parallelogram $ABCD$ is, in fact, a rhombus. Therefore, the area of the gray part equals

$$\operatorname{area}(ABCD) - 4\pi = 2\operatorname{area}(ABD) - 4\pi = a^2\sin(60^\circ) - 4\pi = \frac{a^2\sqrt{3}}{2} - 4\pi,$$

where a is the side of the rhombus. It remains to find a.

Let K and L be the points where the two circles are tangent to the side AD, and P, Q be the centers of the circles (see the figure). Then $a = |AK| + |KL| + |LD|$. Note that $|KL| = |PQ| = 2$. Also $\angle KAP = 30^\circ$ and $\angle LDP = 60^\circ$. Therefore,

$$|AK| = \frac{1}{\tan(30^\circ)} = \sqrt{3} \quad \text{and} \quad |LD| = \frac{1}{\tan(60^\circ)} = \frac{1}{\sqrt{3}}.$$

We obtain $a = \sqrt{3} + 2 + \frac{1}{\sqrt{3}} = 2 + \frac{4}{\sqrt{3}}$. Finally, the area equals

$$\frac{(2+\frac{4}{\sqrt{3}})^2\sqrt{3}}{2} - 4\pi = 2\left(1 + \frac{4}{\sqrt{3}} + \frac{4}{3}\right)\sqrt{3} - 4\pi = 8 + \frac{14}{\sqrt{3}} - 4\pi.$$

(7) Leah and Barbara play a game. There are several piles of pebbles on the table. On every move it is allowed to split any pile into two new piles in such a way that after the move no two piles have the same number of pebbles. The person who cannot make a move looses. For example, there are two piles of 4 and 5 pebbles, respectively. A legal move would be to split the first one into 1 and 3 pebbles, so the result is three piles of 1, 3, and 5 pebbles, respectively. After that no moves are possible.

Leah puts two piles of 5 and 11 pebbles, respectively, on the table. What move must Barbara make to guarantee winning?

Solution: After playing this game for a little while you realize that to win Barbara has to split the 5: either into $\{1,4\}$ or into $\{2,3\}$. Indeed, if Barbara splits the 11 then there are four different positions: $\{1,5,10\}$ or $\{2,5,9\}$ or $\{3,5,8\}$ or $\{4,5,7\}$. In the second case Leah can make $\{1,2,5,8\}$ and Barbara loses. In the other three cases Leah can make $\{1,3,5,7\}$ after which Barbara also loses.

It remains to check that ether one of Barbara's possible moves $\{1,4,11\}$ or $\{2,3,11\}$ leads to winning. Indeed, in either case Leah is forced to split the 11. If Barbara's move is $\{1,4,11\}$, then there are three possible moves Leah can make: $\{1,2,4,9\}$, $\{1,3,4,8\}$ and $\{1,4,5,6\}$. In each of these situations Barbara can make $\{1,2,3,4,6\}$ and she wins!

If Barbara's move is $\{2,3,11\}$ then again there are only three possible moves for Leah: $\{1,2,3,10\}$, $\{2,3,4,7\}$ and $\{2,3,5,6\}$. Just as before, in each of these situations Barbara makes $\{1,2,3,4,6\}$ and wins!

(8) Construct a polynomial $p(x)$ of degree 2009 with integer coefficients and such that $p(1)=1$ and $p(n)=0$ for some positive integer n. (Hint: Try to understand what n can be equal to.)

Solution: We first try to figure out possible values of n. We know that $p(1)=1$, thus, $x=1$ is a root of $p(x)-1$. This implies that $p(x)-1$ factors:

$$p(x)-1=(x-1)g(x),$$

for some polynomial $g(x)$ with integer coefficients. Now plugging $x=n$ in this equation we obtain $-1=(n-1)g(n)$. Since both $(n-1)$ and $g(n)$ are integers, their product can equal -1 only when each of them is 1 or -1. Therefore $n=2$ (remember n must be positive).

Now we need to construct a polynomial of degree 2009 with $p(1)=1$ and $p(2)=0$. We will do it for an arbitrary degree n. We'll assume $p(x)$ has leading coefficient 1, so its general form is

$$p(x)=x^n+\cdots+c_1x+c_0$$

We have $p(2)=2^n+\cdots+c_1\cdot 2+c_0$ and $p(1)=1+\cdots+c_1+c_0$. If we make all the coefficients between x^n and c_1x zero, the remaining two coefficients must satisfy

$$2^n+2c_1+c_0=0 \quad\text{and}\quad 1+c_1+c_0=1.$$

This is a simple linear system whose solution is $c_1=-2^n$ and $c_0=2^n$. Therefore our polynomial is

$$p(x)=x^n-2^nx+2^n.$$

It remains to put $n=2009$ to get the answer for the original question. Notice that we can construct infinitely many such polynomials if we allow also non-zero values for the coefficients between x^n and c_1x.