Fractals, Exemplified by the Koch Snowflake

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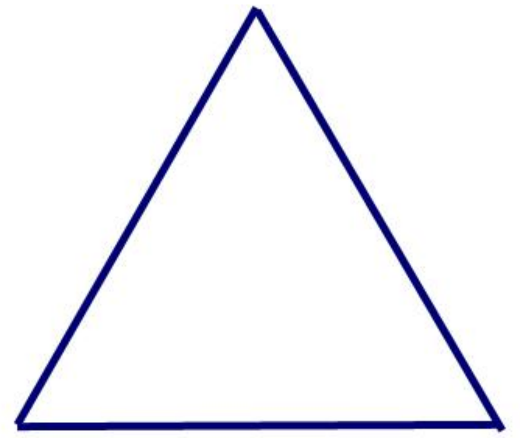
What is a Koch Snowflake?

The Koch Snowflake is fractal formed by the following procedure. Starting with an equallateral triangle, you take each line segment and divide it into thirds. You remove the middle third, but replace it with two line segments of equal length that form another equilateral triangle. On the side of the original triangle. This process is repeated an infinite amount of times. The first few itterations are shown at right.

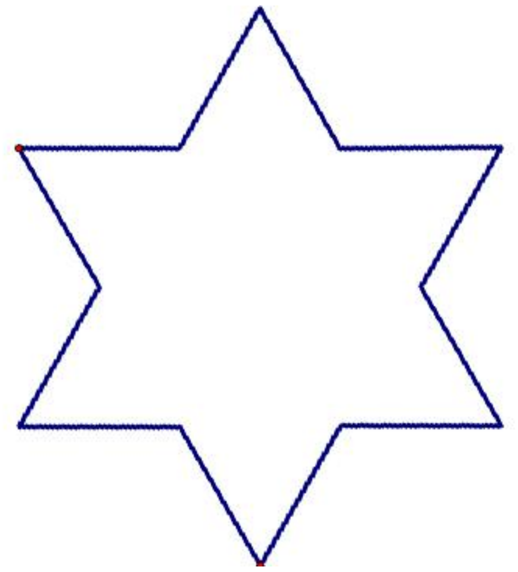
What is a Fractal?

A fractal is a geometric shape that is a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole.

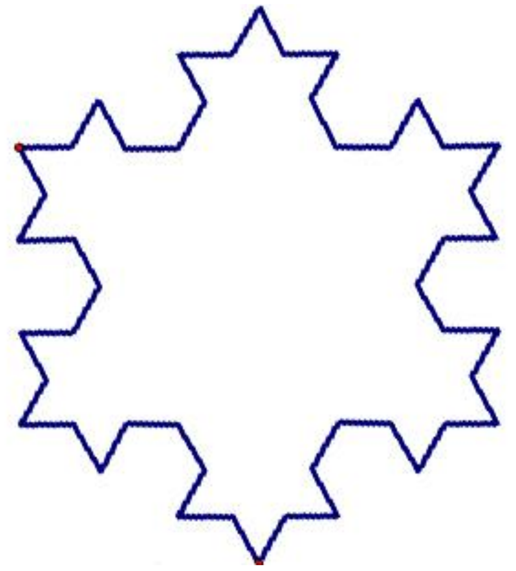
Koch Snowflake picture



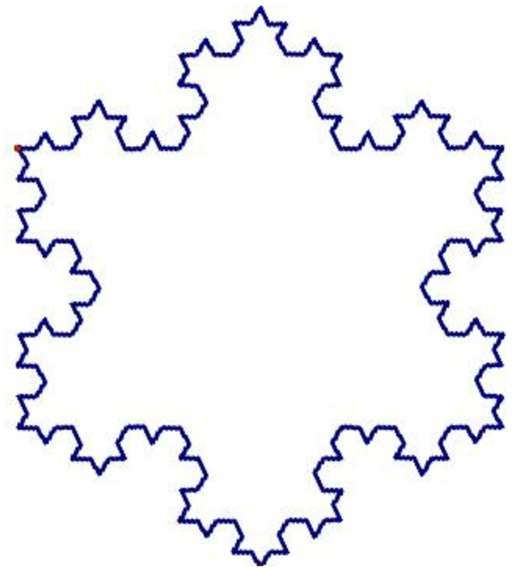
Original Triangle



First Iteration



Second Iteration



Third Iteration

Properties of a Koch Snowflake

Finite Area

If we start with a triangle with sides of length *s*, then the area of the original equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, we shall call this A_0. The triangles added in each iteration are one ninth the area of the previous iteration, because each side is one third the length of the previous side length. The first iteration adds three triangles, while all subsequent iterations add four times the triangles of the previous iteration. This gives us the generic formula $A_{n+1} = A_n + \frac{3\cdot 4^{n-1}}{9^n}A_0$, where $n \geq 1$. We can also define the area after the first iteration is $A_1 = A_0 + 3\cdot\frac{1}{9}\cdot A_0 = A_0 + \frac{1}{3}A_0 = \frac{4}{3}A_0$. Substituting A_1 into the generic formula gives us

$$A_n = \tfrac{4}{3}A_0 + \sum_{k=2}^{n}\frac{3\cdot 4^{n-1}}{9^n}A_0$$

$$A_{n+1} = \tfrac{4}{3}A_0 + \sum_{k=1}^{n}\frac{3\cdot 4^{n}}{9^{n+1}}A_0$$

$$A_{n+1} = \tfrac{4}{3}A_0 + \sum_{k=1}^{n}\frac{3\cdot 4\cdot 4^{n-1}}{81\cdot 9^{n-1}}A_0$$

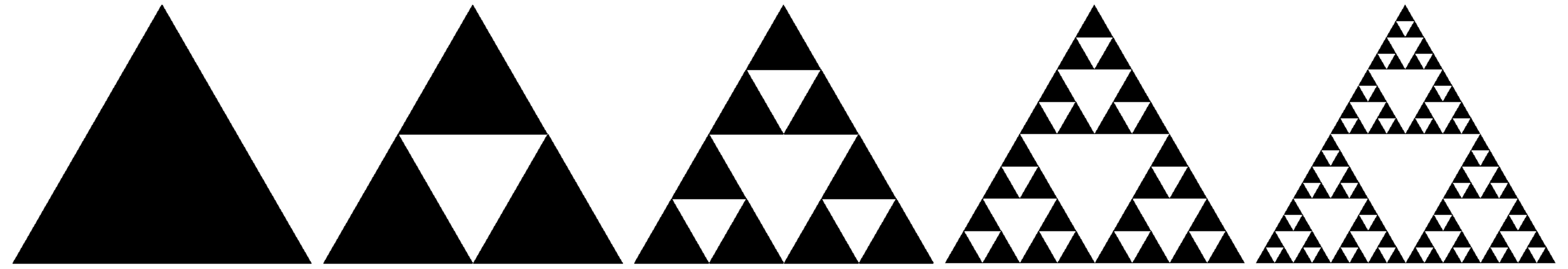
$$A_{n+1} = \tfrac{4}{3}A_0 + \tfrac{4}{27}A_0\sum_{k=1}^{n}\left(\frac{4}{9}\right)^{n-1}$$

The series $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ $\quad if\ |r| < 1$. Therefore, if we take the limit as n approaches infinity, the we get $\lim_{n\to\infty} A_{n+1} = \tfrac{4}{3}A_0 + \tfrac{4}{27}A_0\sum_{k=1}^{n}\left(\frac{4}{9}\right)^{n-1} = \tfrac{4}{3}A_0 + \tfrac{4}{27}A_0\left(\frac{1}{1-\frac{4}{9}}\right) = \tfrac{4}{3}A_0 + \tfrac{4}{27}A_0\left(\frac{1}{\frac{5}{9}}\right) = \tfrac{4}{3}A_0 + \tfrac{4}{27}\cdot\tfrac{9}{5}A_0 = \tfrac{4}{3}A_0 + \tfrac{4}{15}A_0 = \tfrac{8}{5}A_0$

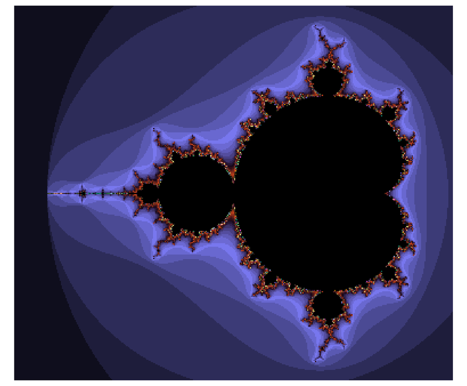
Infinite Perimiter

The Koch Snowflake is a figure with an infinite perimeter but a finite area. We'll start by examining the perimiter. In each iteration, you remove one third of each line segment and replace it with two line segments of equal length, which means that each iteration has a perimiter 4/3 the perimiter of the previous iteration. The area could therefore be described by the equation $P_n = P_0\cdot(4/3)^n$, where n is the number of iterations. Because this is a simple geometric series and r is greater than one, the limit of this series diverges and approaches to infinity as n goes to infinity. The area however is a little more difficult to

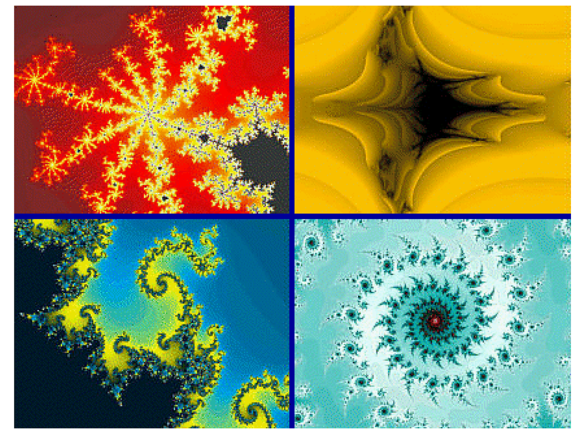
Other Fractals



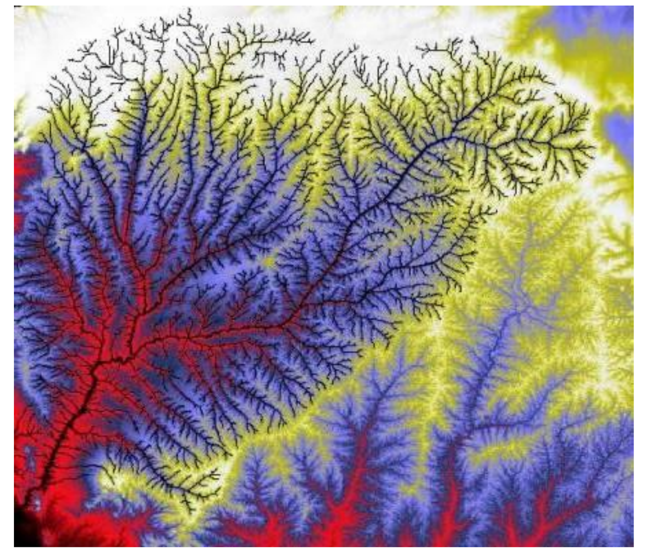
The Sierpinski Triangle is a fractal with an infinite perimeter but an area that that converges to zero. This is because each iteration has ¾ the area of the previous iteration.



The Mandelbrot Fractal



This broccoli is an example of a fractal found in nature.



A river in topographic view