SQUARES INSCRIBED IN CURVES

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Abstract

A mathematical conjecture from Victor Klee's and Stan Wagon's Old and New Unsolved Problems: In Plane Geometry and Number Theory asks "Does every simple closed curve in the plane contain all four vertices of some square?" Here, we are observing smooth curves, such as circles and ellipses, and curves with vertices such as triangles and other polygons. By discovering an algorithm for finding the coordinates and side lengths of a square inside both triangles and ellipses, we can observe the square inscribed in these curves and we can determine if the inscribed square is unique or not.

Square inscribed in an ellipse

The equation for an arbitrary ellipse has the following equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

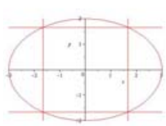
The coordinates for a square centered about the origin are +/-(c,c). Since x and y are the coordinates for the ellipse, we plug in (c,c) for the vertices of the square on the ellipse.

$$\frac{c^2}{a^2} + \frac{c^2}{b^2} = 1$$

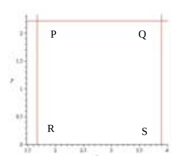
$$c^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 1$$

$$c = \frac{1}{\left(\frac{+}{-}\right)\text{sqrt}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}$$

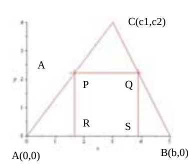
The equation for c solves for the vertices on the ellipse



Square inscribed in a triangle



We define a square with the following vertices



We define the triangle ABC and the square above inscribed in it.

The slope of the segment AC is

$$\frac{c_2}{c_1}$$

The slope of the segment BC is

$$-\frac{c_2}{(b - c_1)}$$

Using the point-slope form the equations for AC and BC are as follows

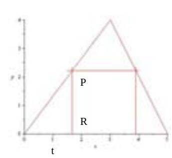
AC

y=($\frac{c_2}{c_1}$)x

BC

y= $-\frac{c_2}{b - c_1}\cdot(x - b)$

The x coordinate for P and R is defined as t in the picture below



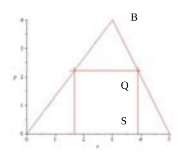
Therefore

R=(t,0)

Since P lies on y=($\frac{c_2}{c_1}$)x, the y-coordinate for P= ($\frac{c_2}{c_1}$)t

so

P=(t, $\frac{c_2}{c_1}$)t)



In order to find the x-coordinate for Q and S we can solve for x using the equation for BC since Q lies on BC and the y-coordinate for Q equals the y-coordinate for P

y= $-\frac{c_2}{b - c_1}\cdot x - b$

y= $\left(-\frac{c_2}{b} - c_1\right)\cdot x + \frac{b\cdot c_2}{b - c_1}$

x= $-\left(y - \left(\frac{b\cdot c_2}{b - c_1}\right)\right)\cdot\frac{(b - c_1)}{c_2}$

The y-coordinate for P=(($\frac{c_2}{c_1}$)t) so

x= $-\left(\left(\frac{c_2}{c_1}\right)\cdot t - \left(\frac{b\cdot c_2}{b - c_1}\right)\right)\cdot\frac{(b - c_1)}{c_2}$

Therefore

Q=($-\left(\left(\frac{c_2}{c_1}\right)\cdot t - \left(\frac{b\cdot c_2}{b - c_1}\right)\right)\cdot\frac{(b - c_1)}{c_2}$, ($\frac{c_2}{c_1}$)t)

S=($-\left(\left(\frac{c_2}{c_1}\right)\cdot t - \left(\frac{b\cdot c_2}{b - c_1}\right)\right)\cdot\frac{(b - c_1)}{c_2}$,0)

In conclusion

The vertices for the square PQRS inscribed in any triangle are as follows when translated and rotated with the longest side on the x-axis

P=(t, ($\frac{c_2}{c_1}$)t)

Q=($-\left(\frac{c_2}{c_1}\cdot t - \frac{(b\cdot c_2)}{b - c_1}\right)\cdot\left(\frac{(b - c_1)}{c_2}\right)$, ($\frac{c_2}{c_1}$)t)

R=($-\left(\frac{c_2}{c_1}\cdot t - \frac{(b\cdot c_2)}{b - c_1}\right)\cdot\left(\frac{(b - c_1)}{c_2}\right)$,0)

S=($-\left(\frac{c_2}{c_1}\cdot t - \frac{(b\cdot c_2)}{b - c_1}\right)\cdot\left(\frac{(b - c_1)}{c_2}\right)$,0)

Reference

Klee, Victor and Wagon, Stan. Old and New Unsolved Problems in Plane geometry and Number Theory. "Inscribed Squares" p. 58-65. The Mathematical Association of America,1991.

Acknowledgements

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