The Brachistochrone Challenge

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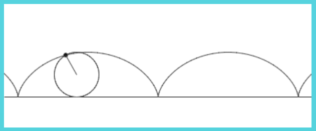
Abstract

The Brachistochrone problem involves finding a curve from one point to another down which a bead will descend without friction and with only the acceleration caused by gravity in the least amount of time. In order to accomplish this, we developed a function to solve for this travel time. Using this function, we tested various equations to determine which took the least amount of time to follow the path. Using the calculus of variations, we found that the cycloid is the curve which took the least amount of time and we then solved for exactly how long it took. The solution to the problem, the cycloid, is the path traced out by a point on the radius of a rolling disc.



Brachistochrone Curve

Motivation

The Brachistochrone problem was posed by Johann Bernoulli to his fellow mathematicians in 1696. It was one of the earliest calculus of variations problems. The solution, which was a segment of a cycloid curve, was found by Leibniz, L'Hospital, Newton, and the two Bernoulli brothers. We wanted to solve this challenge ourselves by experimenting with functions to find the fastest curve.

Methods

The software we used to obtain graphs and travel times for various functions is Maple. To achieve the result of a cycloid for the solution to the problem, the Calculus of Variations was used.

Time Formula

$$t_{12} = \int_{P1}^{P2} \frac{ds}{v}$$

The time to travel from a point P1 to another point P2 is given by the integral; s is arc length and v is speed.

$$\frac{1}{2}mv^2 = mgy$$

$$v = \sqrt{2gy}$$

The speed at any point is given by a simple application of conservation of energy equating kinetic energy to gravitational potential energy.

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx$$

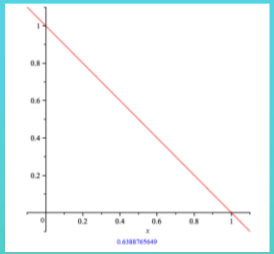
Plug this into the time formula with the identity seen to the left

$$t_{12} = \int_{P1}^{P2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \int_{P1}^{P2} \sqrt{\frac{1+y'^2}{2gy}}\,dx$$

The time formula for a curve.

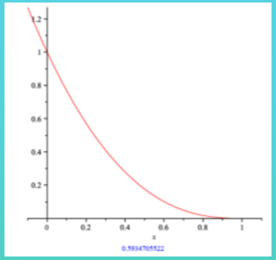
Experiments

F(x) = 1 – X



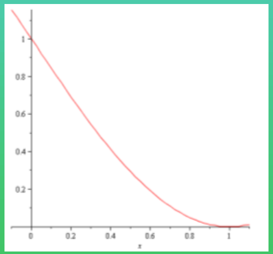
T= .6389s

F(x) = (1 – X)^3



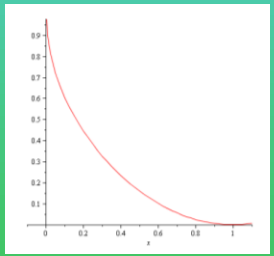
T=.5935s

F(x) = 1-sin(πx/2)



T=.6029s

F(x) = 1-(sin(π x/2)^1/2)



T=.5893s

Brachistochrone Solution

$$f = (1+y'^2)^{\left(\frac{1}{2}\right)} \cdot (2gy)^{\left(-\frac{1}{2}\right)}$$

The time function can be varied to this

$$f - y'\frac{\partial f}{\partial y'} = C$$

We can immediately use the Beltrami identity

$$\frac{1}{\sqrt{2gy}\cdot\sqrt{1+y'^2}} = C$$

Calculating and subtracting $y'(\frac{\partial}{\partial y'} f)$ from f, and simplifying then gives

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right)y = \frac{1}{2gC^2} = k^2$$

Squaring both sides and rearranging slightly results in this

$$x = \frac{k^2}{2}(\theta - \sin\theta)$$

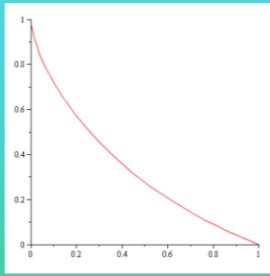
Where the square of the old constant C has been expressed in terms of a new (positive) constant k^2. This equation is solved by the parametric equations

$$y = \frac{k^2}{2}(1 - \cos\theta)$$

Which are--lo and behold--the equations of a cycloid.

Conclusion

Using both experimental methods and the calculus of variations, we concluded that the cycloid was the appropriate curve that takes the least amount of time. By analyzing the other functions we tested, it was concluded that the smoother curves had shorter travel times. To better understand the steps taken to determine that the solution to the problem is a cycloid, we may study the calculus of variations more thoroughly.



T=.5832s

Acknowledgements

Weisstein, Eric W. "Brachistochrone Problem." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/BrachistochroneProblem.html

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